

Tunneling Effect Near Weakly Isolated Horizon

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The tunneling effect near a weakly isolated horizon (WIH) has been studied. By applying the null geodesic method of Parikh and Wilczek and Hamilton-Jacobi method of Angheben et al. to a weakly isolated horizon, we recover the semiclassical emission rate in the tunneling process. We show that the tunneling effect exists in a wide class of spacetimes admitting weakly isolated horizons. The general thermodynamic nature of WIH is then confirmed.

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I. INTRODUCTION

Since the discovery of the first exact solution of Einstein equations, studying properties of black hole is always a highlight of gravitational physics. Laws of black hole mechanics and the famous Hawking radiation reveal deep connections between classical general relativity, quantum physics, and statistical mechanics. It is surprising that all stationary black hole can be described by several elegant laws. Furthermore, black holes are playing a major role in relativistic astrophysics, providing mechanisms to fuel the most powerful engines in the cosmos. In experimental areas, black holes are also important sources for gravitational wave detection. Unfortunately, the classical definition of black hole can not satisfy the requirements of practical research[13]. We need a “quasi-local” definition for black hole. In the last decade, much work has been done in this area[1]-[13]. In contrast to event horizons, the quasi-local treatment doesn’t require the global description of spacetime. Therefore, black hole mechanical laws can be generalized to horizons while ADM quantities are replaced by local defined quantities. A natural question is whether these mechanical laws have thermal dynamical explanation. For stationary black holes, the answer is positive due to the discovery of Hawking radiation[14, 15]. Generalization to isolated horizons is not straightforward because Hawking’s method requires the knowledge of the global geometry in a space-time, not just the geometry near a horizon.

Recently, a new technique concerning Hawking radiation has been developed by Wilczek and his colleagues[16, 17, 18, 19]. By treating Hawking radiation as a tunneling process near the Schwarzschild horizon and using WKB approximation, a semi-classical emission rate

was derived as

$$\Gamma \sim e^{\Delta S_{B-H}}, \quad (1)$$

where ΔS_{B-H} is the change of the Bekenstein-Hawking entropy. Although this result is not exactly the thermal spectrum as first discovered by Hawking, it is consistent with the conservation of energy during the process. This method has been successfully extended to some other types of black holes[20, 21, 22]. All these calculations rely on specific forms of space-time. Since the tunneling effect only concerns the physics around a horizon, it is natural to guess that tunneling effect could be a general phenomenon for all isolated horizons. The isolated horizon theory developed by Ashtekar *et al.* provides a perfect tool to explore this issue. By investigating the outgoing null geodesics near the horizon and applying the first law of the WIH, we recover Eq. (1) for the WIH. The WIH contains the essential features of a black hole, including mass, charge and angular momentum. Thus, our work generalizes the result of the tunneling effect from stationary black holes to a wide class of spacetimes that admit WIHs.

Angheben et al.[20] reconsidered this problem using Hamilton-Jacobi ansatz. The Hamilton-Jacobi provides an alternative derivation of the tunneling effect. It is also a local method involving only an infinitesimal region around the horizon. Similarly, we apply this approach to a WIH, without the knowledge of the spacetime metric, we find the emission rate agrees with Eq. (1). Therefore, it confirms that the Hawking-like radiation exists in very general background.

This paper is organized as follows. In section II, we first give a brief review of Ashtekar’s weakly isolated horizon theory. Technical details can be found in ref.[13] and references therein. In relation to WIH, we introduce the Bondi-like coordinates and calculate the outgoing null geodesic using Newman-Penrose formalism. Section III contains the calculations on the tunneling effect. In section III A and III B, Wilczek’s null geodesic method and Angheben’s Hamilton-Jacobi method are employed respectively to study the tunneling effect near a WIH.

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By using the result in section II, we find that both approaches lead to the desired spectrum Eq. (1). In section III C, we discuss the role of coordinates in the calculations. The master equation in the Parikh-Wilczek method [17] apparently depends on the choice of coordinates. The results in III A apparently depend on the Bondi-like coordinate system chosen in section II. In section III C, we analyze the freedom of the coordinates that keeps the results invariant. We show that what is relevant in the calculation is not the coordinates, but the vector field approaching the horizon. Finally, we make some concluding remarks in section IV.

II. GEOMETRY OF WEAKLY ISOLATED HORIZON

Base on works by Ashtekar and other authors[7]-[13], the weakly isolated horizon is defined by

Definition 1 Let (M, g) be a space-time. \mathcal{H} is a 3-dim null hyper-surface in M and l^a is the tangent vector field of the generator of \mathcal{H} . \mathcal{H} is said to be a **weakly isolated horizon (WIH)**, if

- 1) \mathcal{H} has the topology of $S^2 \times \mathbb{R}$,
- 2) The expansion of the null generator of \mathcal{H} is zero, i.e. $\Theta_l = 0$ on \mathcal{H} ,
- 3) $T_{ab}v^b$ is future causal for any future causal vector v^a and Einstein equation holds in a neighborhood of \mathcal{H} ,
- 4) $[\mathcal{L}_l, D_a]l^b = 0$ on \mathcal{H} , where D_a is the induced covariant derivative on \mathcal{H} .

Since we are going to study the null geodesic behavior around the horizon, we need non-singular coordinates to describe the geometry through the horizon. For the Schwarzschild black hole, Parikh and Wilczek [17] chose the Painlevé coordinates. For the WIH, a convenient choice is the Bondi-like coordinates [23]. Such coordinates have been used to prove the local existence of WIH [24]. The coordinates are constructed as following: The tangent vector of null generator of \mathcal{H} is l^a . Another real null vector field is n^a . The foliation of \mathcal{H} gives us the natural coordinates (θ, ϕ) . Using a parameter u of l^a and Lie dragging (θ, ϕ) along each generator of \mathcal{H} , we have coordinates (u, θ, ϕ) on \mathcal{H} . Choose the affine parameter r of n^a as the forth coordinate, we get the Bondi-like coordinates (u, r, θ, ϕ) near the horizon. In these coordinates, we choose the null tetrad as

$$\begin{aligned} l^a &= \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X \frac{\partial}{\partial \zeta} + \bar{X} \frac{\partial}{\partial \bar{\zeta}}, \\ n^a &= \frac{\partial}{\partial r}, \\ m^a &= \omega \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial \zeta} + \xi^4 \frac{\partial}{\partial \bar{\zeta}}, \end{aligned}$$

$$\bar{m}^a = \bar{\omega} \frac{\partial}{\partial r} + \bar{\xi}^3 \frac{\partial}{\partial \zeta} + \bar{\xi}^4 \frac{\partial}{\partial \bar{\zeta}}, \quad (2)$$

where $U = X = \omega = 0$ on \mathcal{H} (following the notation in ref.[13], equalities restricted to \mathcal{H} will be denoted by " $\hat{=}$ "), $\zeta = e^{i\phi} \cot \frac{\theta}{2}$. In words of Newman-Penrose formalism, we also require the above tetrad satisfy following gauge,

$$\begin{aligned} \nu = \gamma = \tau = \alpha + \bar{\beta} - \pi = \mu - \bar{\mu} &= 0, \\ \varepsilon - \bar{\varepsilon} &\hat{=} \kappa \hat{=} 0, \end{aligned} \quad (3)$$

which means these tetrad vectors are parallelly transported along n^a in space-time and Lie drag-free along l^a on \mathcal{H} . Furthermore, the forth requirement in the definition of WIH implies there exists a one form ω_a on \mathcal{H} such that $D_a l^b \hat{=} \omega_a l^b$ and $\mathcal{L}_l \omega_a \hat{=} 0$. In terms of the Newman-Penrose formalism, ω_a can be expressed as $\omega_a = -(\varepsilon + \bar{\varepsilon})n_a + (\alpha + \bar{\beta})\bar{m}_a + (\bar{\alpha} + \beta)m_a$. The above equation means $(\varepsilon + \bar{\varepsilon})$ is constant on \mathcal{H} . In such a space-time, the out-going null geodesic is described by l^a . We have known that $U \hat{=} 0$. Commutative relation of l^a and n^a tells us that

$$\frac{\partial U}{\partial r} = (\varepsilon + \bar{\varepsilon}) - \bar{\pi}\bar{\omega} - \pi\omega, \quad (4)$$

which means $\frac{\partial U}{\partial r} \hat{=} (\varepsilon + \bar{\varepsilon})$. Then the behavior of function U near \mathcal{H} is

$$U = (\varepsilon + \bar{\varepsilon})r + o(r). \quad (5)$$

Based on the discussion in ref.[13], not any choice of time direction can give a Hamiltonian evolution. In other words, only some suitably chosen time direction can lead to a well-defined horizon mass. In ref.[13], Ashtekar *et al.* gave a canonical way to choose the time direction t^a for a WIH. Compared with the Schwarzschild case [17], the parameter of t^a takes the place of the Killing time. Using the tetrad in Eq. (2), We find the time derivative of r along the outgoing geodesic

$$\dot{r} = \frac{du}{dt} \frac{dr}{du} = (B_t + o(1))U = B_t(\varepsilon + \bar{\varepsilon})r + o(r). \quad (6)$$

By definition, the surface gravity of \mathcal{H} is $\kappa_t := B_t l^a \omega_a = B_t(\varepsilon + \bar{\varepsilon})$. Because $B_t(\varepsilon + \bar{\varepsilon})$ is constant on \mathcal{H} , the zeroth law of black hole mechanics is valid for WIH.

With the canonical time direction t^a , Ashtekar *et al.*[13] showed that the following relation holds on \mathcal{H} .

$$\delta M_{\mathcal{H}}^{(t)} = \frac{\kappa_t}{8\pi} \delta a_{\mathcal{H}} + \Omega_t \delta J_{\mathcal{H}}, \quad (7)$$

where $M_{\mathcal{H}}^{(t)}$ is the horizon mass, $a_{\mathcal{H}}$ is the area of the cross section of WIH, Ω_t is the angular velocity of the horizon and $J_{\mathcal{H}} = -\frac{1}{8\pi} \oint_S (\omega_a \psi^a) dS$ is the angular momentum. This is the first law of black hole mechanics for WIH, which is a generalization of the first law for stationary black holes.

III. TUNNELING EFFECT NEAR WEAKLY ISOLATED HORIZON

A. Null geodesic method

In this section, we consider the tunneling effect near a weakly isolated horizon. We apply Parikh and Wilczek's semi-classical method [17, 18, 19] to a WIH. We shall use the Bondi-like coordinates introduced in section II. It has been demonstrated[17] that the WKB approximation is justified near a horizon. From the WKB formula, the emission rate Γ can be expressed as

$$\Gamma \sim \exp(-2ImS). \quad (8)$$

If the positive energy particle goes outwards, from r_{in} to r_{out} , the imaginary part of action is given by

$$\begin{aligned} ImS &= Im \int_{r_{in}}^{r_{out}} p_r dr, \\ &= Im \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r dr. \end{aligned} \quad (9)$$

From the Hamiltonian equation, p_r can be expressed as

$$dp_r = \frac{dH}{\dot{r}} \quad (10)$$

With the help of Eq. (10) and Eq.(6), the imaginary part of the action becomes

$$\begin{aligned} ImS &= Im \int_M^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{dH}{\dot{r}} dr \\ &= Im \int_M^{M-\omega} \int_{r_{in}}^{r_{out}} \frac{dr}{B_t(\varepsilon + \bar{\varepsilon})r + o(r)} dH \\ &= \pi \int \frac{dH}{B_t(\varepsilon + \bar{\varepsilon})}. \end{aligned} \quad (11)$$

Following the argument in [16, 17], we fix the total mass of the space-time and vary the black hole mass. In section II, the general form of the first law is given by Eq.(7). For the spherically symmetric case ($\Omega_t = 0$), we can express dH in terms of the surface gravity κ_t and the horizon area $a_{\mathcal{H}}$,

$$dH = -dM_{\mathcal{H}}^{(t)} = -\frac{\kappa_t}{8\pi} da_{\mathcal{H}} = -\frac{\kappa_t}{2\pi} d\left(\frac{a_{\mathcal{H}}}{4}\right). \quad (12)$$

The minus sign in above equation comes from the conservation of energy. Then we have

$$\begin{aligned} ImS &= \pi \int \frac{dH}{B_t(\varepsilon + \bar{\varepsilon})} \\ &= -\pi \int \frac{\kappa_t}{2\pi B_t(\varepsilon + \bar{\varepsilon})} d\left(\frac{a_{\mathcal{H}}}{4}\right) \\ &= -\frac{1}{2} \Delta\left(\frac{a_{\mathcal{H}}}{4}\right), \end{aligned} \quad (13)$$

and the emission rate Γ is

$$\Gamma \sim \exp(-2ImS) = \exp(\Delta\frac{a_{\mathcal{H}}}{4}) = \exp(\Delta S_{B-H}) \quad (14)$$

This gives the thermal spectrum of radiation, which is in agreement with Eq. (1).

For an axial symmetric horizon, the first law becomes[10, 13]

$$dM_{\mathcal{H}} = \frac{\kappa_t}{8\pi} da_{\mathcal{H}} + \Omega_t dJ_{\mathcal{H}}. \quad (15)$$

Using the formula in ref.[22] and the first law (7), we have

$$\begin{aligned} ImS &= Im \int (L - P_{\phi} \dot{\phi}) dt \\ &= Im \int [P_r - \frac{P_{\phi} \dot{\phi}}{\dot{r}}] dr \\ &= Im \left(\int \int [dP_r - \frac{\dot{\phi} dP_{\phi}}{\dot{r}}] dr \right) \\ &= Im \left(\int \frac{dH - \dot{\phi} dP_{\phi}}{\dot{r}} dr \right) \\ &= Im \left(\int \frac{-dM_{\mathcal{H}} + \Omega_t dJ_{\mathcal{H}}}{\dot{r}} dr \right) \\ &= -\frac{1}{2} \Delta\left(\frac{a_{\mathcal{H}}}{4}\right), \end{aligned} \quad (16)$$

where the conservation of angular momentum has been used in the fifth equality. Eq. (16) gives the same emission rate near a rotating horizon. Therefore, we have proven that the semiclassical emission rate in form of Eq. (1) holds for a general WIH.

B. Hamilton-Jacobi method

In this section, we use the Hamilton-Jacobi method to re-calculate the Hawking radiation near a WIH. This method has been used in stationary space-times[20, 21]. We need first modify Bondi-like coordinates (t, r, θ, ϕ) . In general, we can always write t^a as $t^a \doteq B_t l^a - X_0^A \left(\frac{\partial}{\partial x^A}\right)^a$, where $A, B = 3, 4$, $X_0^3 = B_t X|_{\mathcal{H}}$, $X_0^4 = B_t \bar{X}|_{\mathcal{H}}$, $x^3 = \zeta$, $x^4 = \bar{\zeta}$. The modified Bondi coordinates are constructed in following way: Instead of using u , we use parameter of t^a as the time coordinate; the coordinates (θ, ϕ) are still obtained from a foliation of \mathcal{H} and extended to \mathcal{H} by Lie-dragged along t^a ; coordinate r is chosen such that $n^a = f \left(\frac{\partial}{\partial r}\right)^a$. Under these coordinates, the only different gauge is that X is constant on \mathcal{H} . In these coordinates, we write the tetrad as

$$\begin{aligned} l^a &= \frac{1}{B_t} \frac{\partial}{\partial t} + U \frac{\partial}{\partial r} + X \frac{\partial}{\partial \zeta} + \bar{X} \frac{\partial}{\partial \bar{\zeta}}, \\ n^a &= f \frac{\partial}{\partial r}, \end{aligned}$$

$$\begin{aligned}
m^a &= \omega \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial \zeta} + \xi^4 \frac{\partial}{\partial \bar{\zeta}}, \\
\bar{m}^a &= \bar{\omega} \frac{\partial}{\partial r} + \bar{\xi}^3 \frac{\partial}{\partial \bar{\zeta}} + \bar{\xi}^4 \frac{\partial}{\partial \zeta}.
\end{aligned} \tag{17}$$

where $B_t X \hat{=} X_0$, $f \hat{=} 1$, $\omega \hat{=} 0$, $U \hat{=} 0$. Then the metric $g^{\mu\nu}$ takes the form

$$\begin{pmatrix} 0 & \frac{f}{B_t} & 0 \\ \frac{f}{B_t} & 2(fU - |\omega|^2) & fX^B - (\bar{\omega}\xi^B + \omega\bar{\xi}^B) \\ 0 & fX^A - (\bar{\omega}\xi^A + \omega\bar{\xi}^A) & -(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B) \end{pmatrix}. \tag{18}$$

In addition, we shall use the same gauge condition (3).

Based on this ansatz, the action function I of an outgoing particle should satisfy following equation :

$$g^{\mu\nu} \partial_\mu I \partial_\nu I + m^2 = 0. \tag{19}$$

Comparing with null geodesic method, we also need to control other components of metric near the horizon. Note that the gauge choice in section II has fixed the metric components on \mathcal{H} . The behavior of metric components near the horizon is controlled by Cartan structure equations,

$$\begin{aligned}
\frac{\partial U}{\partial r} &= (\varepsilon + \bar{\varepsilon}) - \frac{\bar{\pi}\bar{\omega} + \pi\omega}{f} + D \ln f, \\
\frac{\partial X}{\partial r} &= -\frac{\bar{\pi}\bar{\xi}^4 + \pi\xi^3}{f}, \\
\frac{\partial \omega}{\partial r} &= \bar{\pi} - \frac{\bar{\lambda}\bar{\omega}}{f} - \frac{\mu\omega}{f} + \delta \ln f, \\
\frac{\partial \xi^3}{\partial r} &= -\frac{\bar{\lambda}\bar{\xi}^4 + \mu\xi^3}{f}, \\
\frac{\partial \xi^4}{\partial r} &= -\frac{\bar{\lambda}\bar{\xi}^3 + \mu\xi^4}{f},
\end{aligned} \tag{20}$$

where $D := l^a \nabla_a$, $\delta := m^a \nabla_a$. Here we have used the modified Bondi coordinates (t, r, θ, ϕ) . The behavior of metric near horizon is

$$\begin{aligned}
U &= (\varepsilon + \bar{\varepsilon})r + o(r), \\
X &= X_0 + O(r), \\
\xi^3 &= O(1), \\
\xi^4 &= O(1), \\
\omega &= O(r).
\end{aligned} \tag{21}$$

The time derivatives of these quantities are

$$\begin{aligned}
\frac{\partial U}{\partial t} &= o(r), \\
\frac{\partial X}{\partial t} &= O(r), \\
\frac{\partial \omega}{\partial t} &= O(r), \\
\frac{\partial X^A}{\partial t} &= O(r).
\end{aligned} \tag{22}$$

We consider the variable separation solution for I

$$I = V(t) + W(r) + J(x^A). \tag{23}$$

Eq.(19) then becomes

$$\begin{aligned}
&2(fU - |\omega|^2)(W')^2 \\
&+ 2[(fX^A - \bar{\omega}\xi^A - \omega\bar{\xi}^A)J_A + \frac{f}{B_t}\dot{V}]W' \\
&- (\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B)J_A J_B + m^2 = 0,
\end{aligned} \tag{24}$$

where “ \cdot ” means $\frac{\partial}{\partial t}$, “ \cdot ” means $\frac{\partial}{\partial r}$ and $J_A = \frac{\partial J}{\partial x^A}$. Consequently, we have

$$W(r) = \int \frac{-\mathcal{B} + \sqrt{\mathcal{B}^2 - \mathcal{A}\mathcal{C}}}{\mathcal{A}} dr, \tag{25}$$

where

$$\begin{aligned}
\mathcal{A} &= 2(fU - |\omega|^2), \\
\mathcal{B} &= (fX^A - \bar{\omega}\xi^A - \omega\bar{\xi}^A)J_A + \frac{f}{B_t}\dot{V}, \\
\mathcal{C} &= -(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B)J_A J_B + m^2.
\end{aligned}$$

Taking the time derivative on both sides of Eq. (25) gives

$$\begin{aligned}
&2(\dot{f}U + f\dot{U} - \bar{\omega}\dot{\omega} - \omega\dot{\bar{\omega}})(W')^2 \\
&+ 2\left[(\dot{f}X^A + f\dot{X}^A - \dot{\bar{\omega}}\xi^A - \bar{\omega}\dot{\xi}^A - \dot{\omega}\bar{\xi}^A - \omega\dot{\bar{\xi}}^A)J_A \right. \\
&\quad \left. + \frac{\dot{f}}{B_t}\dot{V} + \frac{f}{B_t}\ddot{V}\right]W' \\
&- \frac{\partial}{\partial t}(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B)J_A J_B = 0,
\end{aligned} \tag{26}$$

Because Eq. (26) holds smoothly in the neighborhood of \mathcal{H} , with the help of the asymptotic behavior of the tetrad (21) and (22), we get $\dot{X}^A J_A + \frac{\ddot{V}}{B_t} = 0$. Furthermore, similar derivation shows $\partial_B(X^A J_A) = 0$, i.e. $X^A J_A + \frac{\dot{V}}{B_t} \hat{=} const$. In the axisymmetric case, X_0 is constant on \mathcal{H} [13]. Thus, \dot{V} and J_A are constants. Comparing with the stationary case, $E = -\dot{V}$ is the energy of this particle and J_A is the angular momentum of the particle. The energy condition guarantees $E - B_t X_0^A J_A \geq 0$. With the above results and the asymptotic behavior, we find that integral (25) has a simple pole at horizon. The imaginary part of the action function is determined by this pole. Specifically,

$$Im(I) = Im\left(\int W' dr\right) = \pi \frac{E - B_t X_0^A J_A}{\kappa_t}, \tag{27}$$

where $\kappa_t = B_t(\varepsilon + \bar{\varepsilon})$. Combining with the WKB assumption $\Psi(x) = \exp(iI)$, we get the thermal spectrum $\Gamma \sim \exp[-\beta(E - B_t X_0^A J_A)]$. As we emphasized in section II, Only a canonical time direction t^a can lead to a well-defined horizon mass and angular momentum, as well as the black hole mechanical law. For such an observer, $t^a \hat{=} B_t l^a - \Omega_t \psi^a$, where ψ^a is a Killing vector of

\mathcal{H} and tangent to the leaf S , Ω_t is the angular velocity of the horizon. In this case, we choose X_0^A as $X_0^A = \Omega_t \psi^A$. Because of the conservation of total energy and angular momentum, E is the variation of the energy of black hole and $\psi(J)$ is the variation of the black hole. Combining with the quasi-local black hole mechanical law (7), we obtain $\Gamma \sim \exp(\Delta \frac{a\chi}{4})$.

C. Freedom of coordinates

The calculation in section III A appears to rely on a specific coordinate system. The purpose of this subsection is to investigate how much the result depends on the choice of coordinates. In order to calculate the tunnelling effect, the crucial step is to calculate the residue of $1/\dot{r}$. We consider the coordinates (t, r, θ, ϕ) as defined in section II, where t is the canonical time. It will be sufficient to focus on the following ine element

$$ds^2 = g_{tt}dt^2 + 2g_{tr}dtdr + g_{rr}dr^2 + g_{AB}dx^A dx^B. \quad (28)$$

These coordinates are nonsingular through the horizon. By letting $ds^2 = 0$, \dot{r} , the derivative of r along an outgoing null geodesic, can be easily solved as

$$\dot{r} = \frac{-g_{tr} + \sqrt{g_{tr}^2 - g_{rr}(g_{tt} + g_{AB}\dot{x}^A \dot{x}^B)}}{g_{rr}}. \quad (29)$$

Since $(\frac{\partial}{\partial t})^a$ approaches the horizon, g_{tt} must vanish on the horizon. Since r is chosen to be a constant on the horizon, \dot{r} also vanishes on the horizon. Then we have

$$g_{rr}(g_{tt} + g_{AB}\dot{x}^A \dot{x}^B) \hat{=} 0. \quad (30)$$

Note that $g_{tt} \hat{=} 0$ and the metric is non-degenerate around the horizon. Thus g_{tr} must be non-zero at the horizon. So we can simplify Eq. (29) by Taylor expanding the square root term around g_{tr}^2 . Then we find

$$\frac{1}{\dot{r}} \approx -\frac{2g_{tr}}{g_{tt} + g_{AB}\dot{x}^A \dot{x}^B} \quad (31)$$

It is not difficult to see that the approximation made in Eq. (31) preserves the residue. Then Eq. (31) leads to the desired emission rate as shown in section III A. Now consider the coordinate transformation

$$\begin{aligned} t &= T + f(R) \\ r &= R, \end{aligned} \quad (32)$$

where $f(R)$ is a smooth function around the horizon. Under the new coordinates, the metric becomes

$$\begin{aligned} ds^2 &= g_{tt}dT^2 + [2g_{tr} + g_{tt}f'(r)]dTdR \\ &\quad + [g_{rr} + 2g_{tr}f'(r) + g_{tt}f'(r)^2]dR^2 \\ &\quad + g_{AB}dx^A dx^B, \end{aligned} \quad (33)$$

which gives the radial null geodesic equation

$$\begin{aligned} &[g_{rr} + 2g_{tr}f'(r) + g_{tt}f'(r)^2]\dot{R}^2 \\ &+ [2g_{tr} + g_{tt}f'(r)]\dot{R} + g_{tt} + g_{AB}\dot{x}^A \dot{x}^B = 0, \end{aligned} \quad (34)$$

where $\dot{R} = dR/dT$. The outgoing solution for $\frac{1}{\dot{R}}$ is

$$\frac{1}{\dot{R}} = \frac{2(g_{rr} + 2g_{tr}f'(r) + g_{tt}f'(r)^2)}{-2g_{tr} - g_{tt}f'(r) + \sqrt{(2g_{tr} + g_{tt}f'(r))^2 - 4[g_{rr} + 2g_{tr}f'(r) + g_{tt}f'(r)^2][g_{tt} + g_{AB}\dot{x}^A \dot{x}^B]}}. \quad (35)$$

Note that $g_{tt} = 0$ at the horizon. Since the coordinates are nonsingular, g_{tr} cannot be zero at the horizon. In order to find the residue of $1/\dot{R}$, we Taylor expand the square root around $(2g_{tr} + g_{tt}f'(r))^2$ to order one (terms with higher orders are not relevant to the calculation of residue). Then Eq. (35) becomes

$$\frac{1}{\dot{R}} \approx -\frac{2g_{tr} + g_{tt}f'(r)}{g_{tt} + g_{AB}\dot{x}^A \dot{x}^B}. \quad (36)$$

Noting that g_{tt} vanishes on the horizon and comparing Eq. (36) with Eq. (31), we see that $1/\dot{R}$ has the same residue as $1/\dot{r}$.

Now we are going to show that the result is invariant under a rescaling of r . Consider the following coordinate transformation:

$$\begin{aligned} t &= \tilde{t} \\ r &= h(\tilde{r}), \end{aligned} \quad (37)$$

where \tilde{r} is any smooth function of r satisfying $\tilde{r} = 0$ on the horizon. Similarly, we find

$$\frac{1}{\tilde{r}} = \frac{f'(\tilde{r})}{-\frac{g_{tr}}{g_{rr}} + \sqrt{\left(\frac{g_{tr}}{g_{rr}}\right)^2 - \frac{g_{tt} + g_{AB}\dot{x}^A \dot{x}^B}{g_{rr}}}}$$

$$\approx \frac{2g_{tr}f'(\tilde{r})}{g_{tt} + g_{AB}\dot{x}^A\dot{x}^B}. \quad (38)$$

Suppose the residue of $1/\dot{r}$ at the horizon is α . Then

$$\frac{1}{\dot{r}} \rightarrow \frac{\alpha f'(\tilde{r})}{r} = \frac{\alpha f'(\tilde{r})}{f(\tilde{r})} \approx \frac{\alpha f'(\tilde{r}=0)}{f'(\tilde{r}=0)\tilde{r}} = \frac{\alpha}{\tilde{r}}, \quad (39)$$

which shows that $1/\dot{r}$ and $1/\tilde{r}$ have the same residue on the horizon.

We have shown that there exist two classes of coordinate transformations, Eq. (32) and Eq. (37), that keep the residue invariant in the tunnelling effect. However, the two transformations share a common feature: they both preserve the tangent field $(\frac{\partial}{\partial t})^a$. So we may conclude that the emission rate is determined only by a specific family of orbits approaching the horizon and parameterized by some coordinate time t (which is the Killing time in the Schwarzschild case). All comoving coordinate systems preserving the tangent field $(\frac{\partial}{\partial t})^a$ are equivalent in the sense that they lead to the same emission rate.

Now we keep the orbits fixed on the horizon but vary the orbits off the horizon. This can be realized by the following coordinate transformation

$$\begin{aligned} r &= h(T, R) \\ t &= T. \end{aligned} \quad (40)$$

where $\dot{h} = \frac{\partial h}{\partial T} \neq 0$. Then

$$\left(\frac{\partial}{\partial T}\right)^a = \left(\frac{\partial}{\partial t}\right)^a + \dot{h} \left(\frac{\partial}{\partial r}\right)^a, \quad (41)$$

which shows that $(\frac{\partial}{\partial t})^a$ and $(\frac{\partial}{\partial T})^a$ coincide on the horizon but are different off the horizon. By a similar derivation, we have

$$\begin{aligned} \frac{1}{\dot{R}} &= -\frac{g_{rr}h'}{g_{tr} + g_{rr}\dot{h} - \sqrt{g_{tr}^2 - g_{rr}(g_{tt} + g_{AB}\dot{x}^A\dot{x}^B)}} \\ &\approx \frac{-2g_{tr}h'}{2g_{tr}\dot{h} + g_{tt} + g_{AB}\dot{x}^A\dot{x}^B}. \end{aligned} \quad (42)$$

A derivation similar to Eq. (39) shows that h' in Eq. (42) does not change the residue. However, if $\dot{h} \sim r$ near the horizon, it will cause a change of residue. For $\dot{h} \sim r^n$ ($n \geq 2$), the residue keeps invariant. Since \dot{h} measures the difference of the two families of orbits approaching the horizon (see Eq. (41)), the above results show that when the difference is in the order of r , the residue will change. But if the difference is smaller than r (higher order of r), the residue is unchanged. Thus, if the two families of orbits in Eq. (41) satisfy $\dot{h} \sim r^n$, $n \geq 2$, we

call them equivalent in the sense that they lead to the same residue.

IV. DISCUSSION

The Hawking-like radiation has been derived as a tunneling process near weakly isolated horizons. The null geodesic method and Hamilton-Jacobi ansatz lead to the same result. This indicates that thermal radiation is a generic property of horizon, not only for stationary black holes. The choice of time direction has played a key role in both methods. However, there are subtle differences in the two methods. In the null geodesic method, only the canonical time direction can define the horizon mass and lead to the first law of black hole mechanics. In the Hamilton-Jacobi method, the thermal spectrum exists for any choice of time direction. However, only when the canonical time direction is chosen, can the κ_t term in the expression be interpreted as the Hawking temperature. We also find that different observers will give different temperatures even if they have the same limit curve at the horizon. We find the difference $\partial_t - \partial_{t_c} \sim O(r^2)$ can insure the invariance of the temperature, where t_c is the canonical time given by Ashtekar *et al.*[13]. This result is reasonable because the energy and angular momentum measured by arbitrary observers cannot be directly related to the horizon mass. The canonical time direction corresponds to the Killing observers in Schwarzschild spacetime, where other observers are like the Rindler observers of the Unruh effect.

Our discussion in this paper is confined to vacuum solutions. There is no obvious difficulty to extend the discussion to matter fields, for example, the Maxwell field or other gauge fields. When matter is considered, the field equations can be controlled in similar ways and the first law including matter fields is also known[1]-[13]. The next interesting thing is studying dynamical horizons. Some special cases have been investigated[25]. The general calculation on dynamical horizons will be done in future work.

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